Review of *Basic Proof Theory* (second edition), by A. S. Troelstra and H. Schwichtenberg

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1 Overview

Beweistheorie, or "Proof Theory," was the phrase that David Hilbert used to describe the program by which he hoped to secure the foundations of mathematics. Set forth in the early 1920's, his plan was to represent mathematical reasoning by formal deductive systems, and show, using safe, "finitary," methods, that such reasoning could never lead to contradiction. This particular goal was shown by Gödel to be infeasible. But the more general goal of using formal methods to explore various aspects of mathematical provability, including the relationship between classical and constructive methods in mathematics and the strengths and limitations of various axiomatic frameworks, has proved to be remarkably robust. Today, these goals represent the traditional, *metamathematical* branch of proof theory.

Since Hilbert's time, the subject has expanded in two important respects. First, it has moved well beyond the study of specifically mathematical reasoning. Proof theorists now consider a wide range of deductive systems, designed to model diverse aspects of logical inference; for example, systems of modal logic can be used to model reasoning about possible states of affairs, knowledge, or time, and linear logic provides a means of reasoning about computational resources.

The second change is that now more attention is paid to specific features of the deductive systems themselves. For the Hilbert school, deductive calculi were used primarily as a means of exploring the deductive consequences of various axiomatic theories; from this point of view, the particular choice of a calculus is largely a matter of convenience. In much of modern proof theory, however, deductive systems have become objects of study in their own right. For example, in the branch of proof theory known as *proof complexity*, one aims to determine the efficiency of various systems with respect to various measures proof length, just as the field of computational complexity explores issues of computational efficiency with respect to various measures of complexity.

In the field of *structural proof theory*, the focus is on the particular syntactic representations, axioms, and rules. A central theme is the exploration of transformations, or "reductions," that preserve the validity of a proof. One of the subject's typical goals is to show that proofs in a given calculus can be brought (e.g. via cut-elimination or normalization) into particularly nice canonical forms. There are various reasons for doing so: for example, from a proof in canonical form, it is often possible to extract additional information; and knowing that every proof has a canonical form means that in searching for proofs it is sufficient to search for a canonical representative.

This emphasis on syntax makes structural proof theory a black sheep in the mathematical logic community. Learning to appreciate the beauty of the subject takes some effort, but one is, in the end, rewarded by elegant combinatorial symmetries and theorems that are often striking or unexpected. Moreover, the subject is closely linked to computational applications. The fields of artificial intelligence, hardware and software verification, logic programming, and automated deduction are strongly dependent on logical inference, and any computational task that involves searching for and manipulating proofs in a serious way requires a solid understanding of deductive systems and their properties. There are also substantial interactions with the theory of programming languages. Implicit in the early development of intuitionistic logic, in the hands of Brouwer, Heyting, and Kolmogorov, is the notion that an intuitionistic proof *is* a program, returning a type of evidence that is specified by the proof's conclusion. A formal analysis of the correspondence between proofs and programs gives rise to constructive type theory, which is of foundational importance to the design of functional programming languages. Given the inevitable tensions between pure and applied branches of any discipline, structural proof theory is a remarkably happy marriage of theory and practice.

Basic Proof Theory is a thorough introduction to structural proof theory. It offers a unified and comprehensive account of the core fundamentals of the subject, and, in doing so, it fills a major expository gap in the literature. The book also presents good deal of additional information in a clear and organized manner, with historical notes and references at the end of each chapter. It serves admirably as a standard reference for the subject, and is likely to maintain that role for a number of years.

2 Contents

The book's eleven chapters can be divided neatly into two parts. The first six chapters provide a comprehensive overview of the subject's essentials. The wealth of information in these chapters may overwhelm the novice, but the exposition is crisp and clear, and those making it through these chapters are thereby licensed to proof-theorize. Topics treated include all of the following (not necessarily in the order indicated):

- the three basic versions of propositional and first-order logic, i.e. classical logic, intuitionistic logic, and minimal logic; various treatments of equality; logic with partial (possibly non-denoting) terms; and the introduction of defined function symbols;

- a thorough treatment of various deductive calculi, including axiomatic calculi, natural deduction, and sequent calculi, with numerous variations, and translations between the various formalisms;
- various double-negation translations, which serve to interpret classical logic in intuitionistic or minimal logic;
- cut-elimination, with variations and extensions, and both upper bounds and lower bounds on the increase in proof length;
- applications of the cut-elimination theorem: Herbrand's theorem, the explicit definability and disjunction properties, interpolation theorems, and conservation results;
- proof terms and the Curry-Howard isomorphism; strong normalization and the Church-Rosser property for the simply typed lambda calculus; weak normalization for full first-order logic; and applications of normalization.

The remaining five chapters make up what I am calling the second part, and survey a number of related topics. In each case, the goal is not to provide a comprehensive introduction, but, rather, to convey the flavor of the subject, and indicate some of the interactions with the core ideas and methods of structural proof theory. Topics treated include the following:

- Resolution: the focus is on linear resolution, which is important to logic programming, but there is also a discussion of the more general form of resolution, and the relationships to other deductive calculi;
- Categorical logic: deduction graphs, the relationship between cartesian-closed categories and the simply typed lambda calculus, and some coherence theorems;
- Modal logic: the focus is on propositional S4, with a cut-elimination theorem, and embeddings of intuitionistic logic;
- Linear logic: cut-elimination, embeddings of classical and intuitionistic logic, and proof nets;
- The ordinal analysis of arithmetic;
- Second-order logic, including the second-order lambda calculus, and Girard's proof of strong normalization.

3 Opinions

The preface indicates that the book is intended as a first step beyond standard introductions to logic, providing adequate preparation for understanding contemporary literature and more advanced monographs on the subject. As such, it will appeal to both computer scientists and mathematical logicians, and graduate students in these disciplines (as well as, perhaps, some very advanced undergraduates). But the text is by no means light reading, and calls for a fair amount of mathematical maturity, although no specific mathematical background is required. The proofs are compact, and the motivation behind some of the definitions and lemmata must often be inferred from their ultimate use. Routine syntactic details are frequently left to the reader.

The treatment of the basics of structural proof theory in the first six chapters is quite detailed. For example, both two-sided and one-sided sequent calculi are considered, using either sets or multisets as sequents, and with many variations of the rules of inference. This encyclopedic treatment is sometimes at odds with the pedagogical goals of an introductory text: someone coming to the subject from scratch may well prefer to avoid this breadth and focus on one or two representative variations. In that respect, the book calls for selective reading. The advantage to having the additional information at hand is that the text can grow with the reader's understanding. Furthermore, it allows the authors to include a number of fundamentally important results that are not found in other expository texts, because they are either too new, or folklore, or buried in the technical literature.

If the first part of the book runs the risk of including too much information, the second part runs the risk of treating its topics too lightly. For example, most of the material in the chapter on categorical logic has been around since the 1970's, and the overall treatment does not do justice to the importance of categorical methods in the study of intuitionistic first- and higher-order logic, or in the study of simple and dependent type theory. Similarly, though linear logic plays an important role in the analysis of concurrency, resource-bounded computation, and games, the book's brief presentation is largely unmotivated, and the applications to the analysis of classical and intuitionistic logic given as examples are, by themselves, not very satisfying. Of course, everyone will have his or her own quibbles with other aspects of the presentation. For example, I would have chosen to present the more general form of resolution first, with linear resolution as a special case; I found the motivational introduction in Section 7.1 a little confusing. Those familiar with linear logic may be put off by the authors' nonstandard choice of symbols. And the book's presentation of the ordinal analysis of arithmetic uses an infinitary extension of natural deduction, which is technically less smooth than presentations using an infinitary sequent calculus (such as the second author's article in the *Handbook of Mathematical Logic*).

Taken in isolation, the text does not provide a comprehensive introduction to any of the subjects covered in the last five chapters. But it is unfair to criticize the book for failing to provide an adequate introduction to seven subjects, when it only aims to provide an introduction to one. The survey offers a good starting point for exploring the additional topics from a proof-theoretic point of view, and, for those who come to this book with prior or independent interest in these topics, it offers a good sense of what proof-theoretic analysis can do.

In the preface, the authors make it clear that the two related branches of proof theory, i.e. proof complexity and metamathematical proof theory, are not the focus of attention. They were wise to limit the scope in this way, and *Basic Proof Theory* nicely complements books devoted to these other subjects. Of course, there is a good deal of overlap between the various branches of proof theory, and the book is attentive to related issues. For example, it provides exponential lower bounds on cut-free propositional provability and hyperexponential lower bounds in the first-order case, which are fundamental results in proof complexity, due to Statman and Orevkov; and it presents Gentzen's results on the ordinal analysis of arithmetic, which is equally fundamental to the metamathematical branch of the subject.

In the second edition, substantial additions and revisions have been made, and a number of errors have been corrected. Monographs of this sort are often obnoxiously overpriced, so the authors and publisher are to be commended for keeping this volume pleasantly affordable ($34.95 in the US). I hope that the shortcomings discussed in this section do not obscure the bottom line: this book is a welcome and important contribution to the expository literature on structual proof theory, and anyone whose professional or intellectual interests involve formal deductive systems in any way will want to keep a copy close at hand.

Basic Proof Theory Second Edition A. S. Troelstra and H. Schwichtenberg s i h i. Basic Proof Theory Second Edition A.S. Troelstra University of Amsterdam H. Schwichtenberg University of Munich (CAMBRIDGE 1 UNIVERSITY PRESS.Â Contents Preface ix 1 Introduction 1 1.1 Preliminaries 2 1.2 Simple type theories 10 1.3 Three types of formalism 22 2 N-systems and H-systems 35 2.1 Natural deduction systems 35 2.2 Ni as a term calculus 45 2.3 The relation between C, I and M 48 2.4 Hilbert systems 51 2.5 Notes 55. Editorial Reviews. Review. 'This is a fine book. Any computer scientist with some logical background will benefit from studying it. It is written by two of the experts in the field and comes up to their usual standards of precision and care.' Ray Turner, Computer Journal. Read more. Book Description. This is an introduction to the basic ideas of structural proof theory.Â Troelstra and Schwichtenberg did not think interesting proof theory stops at cut-elimination, or at Gentzen's elaborate "proof" of the consistency of arithmetic using transfinite induction (Tarski claimed this latter item advanced his understanding of the issue "not one epsilon").Â Anyone wanting a first introduction to proof theory will probably find the one by Pohlers a lot more exciting than this one.