

A Beginner's Guide to Noncommutative Geometry

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1 Introduction

What kind of background does a student need in order to start working in Noncommutative Geometry (NCG)? The following text is an attempt to answer this question but I should warn the reader about two things first: 1) the material emphasized here is a kind of backbone to NCG and is not the body of material that constitutes NCG itself; in particular, references to the NCG literature are scarce, 2) by no means am I suggesting here that the students of NCG should stop learning and doing NCG and start learning the background material first. One learns mathematics by doing mathematics, and you pick up more and more items as you go along and as you need them. This way your learning will also be faster and deeper. The stuff emphasized here is rather universal in that it is prepared with a view towards the totality of NCG. To work in each particular area, at least in the beginning, you don't need to know too much about neighboring areas. For general advice, I can't do better than highly recommending Alain Connes' article, *advice to the beginner* [7].

I have divided the background material into four parts: Algebra, Analysis, Geometry and Topology, and Physics. Each section is accompanied by only a very few selected sample references. You should find you own favorites yourself!

2 Algebra

Cyclic (co)homology, K-theory, K-homology, and KK-theory are topological invariants of noncommutative spaces. To understand cyclic (co)homology and its relation with other (co)homology theories, homological algebra is essential. Before the discovery of cyclic cohomology, mathematicians had discovered a whole series of cohomology theories for spaces (such as singular, Čech, de Rham, étale, etc.), and for algebraic structures (like Hochschild cohomology, group cohomology, Lie algebra cohomology). Homological algebra provides the means to compute these theories through the theory of derived functors and studies the links between them. Weibel [40] and Gelfand-Manin [19], are two modern introductions to homological algebra and will give you a good grounding in the subject.

- *(co)chain complexes of vector spaces (and R -modules) and their (co)homology, chain maps, chain homotopy exact sequences, fundamental lemma of homological algebra: the long exact sequence of a short sequence and its connecting homomorphism, resolutions, bar and Koszul resolutions, projective and injective resolutions, comparison theorem of resolutions*
Ext and Tor functors,
Koszul algebras and Koszul duality
group (co)cohomology, Lie algebra (co)homology, Hochschild (co)homology, spectral sequences

Triangulated and derived categories are very much in vogue these days, both in physics (homological mirror symmetry, string theory), in algebraic geometry and homotopy theory, and in noncommutative geometry. If you need to know this, a very good place to start is Gelfand-Manin [19]. Those who are more interested in noncommutative algebraic geometry should learn about Grothendieck's work, on which a good introduction [22] has just appeared.

There are very exciting connections between number theory and noncommutative geometry, starting with the work of Bost and Connes and with Connes' approach to Riemann hypothesis, relating zeta and other L -functions to quantum statistical mechanics (cf. [10] for recent work and full references). The forthcoming book by Connes and Marcolli [12] will contain the latest on this and much more. Matilde Marcolli's book [29] is a nice introduction to

interactions between arithmetic algebraic geometry and NCG. For a modern survey of number theory, which incidentally covers connections with NCG as well, look at Manin and Panchishkin's book [28]. One of the best introductions to algebraic number theory with a modern perspective is Neukirch's [31].

Quantum groups and *Hopf algebras* are now indispensable for noncommutative geometry and its applications. A good starting point to learn about both subjects is Majid's book [27], which points to many references for further reading as well.

3 Analysis

Index theory of elliptic differential operators, K-theory, K-homology and KK-theory, and spectral geometry are the heart of noncommutative geometry. The common idea here is the study of a space and its topological invariants through operators on a Hilbert space. So far it seems this is the only way one can extend ideas of geometry to the noncommutative realm. There is a beautiful synthesis of these ideas in Connes' notion of a *spectral triple*, which among other things captures metric aspects of noncommutative spaces [9]. The basic analysis here is that of operators on a Hilbert space. For this reason functional analysis and the theory of operator algebras are essential for NCG. There is no shortage of good books here but I recommend the books by P. Lax [25], and Reed and Simon [33]. For an introduction to Banach and C^* -algebra techniques check out Douglas [16], or Fillmore [17]. The history of functional analysis is well presented in Dieudonn, from which one can also learn a lot. An elegant survey of von Neumann and C^* -algebras and their place in NCG can be found in Connes' book [9]. See also Takesaki's [38].

- *Hilbert space and its unique place among all Banach space; at a later stage you also need to know about topological vector spaces*
bounded and unbounded operators on a Hilbert space
spectral theory and the spectral theorem
compact, trace class, Hilbert-Schmidt, and p -summable operators; ideals of operators
Fredholm operators and Fredholm index, abstract index theory
Banach algebras, Gelfand's theory of commutative Banach algebras
theorems of Gelfand-Naimark, and Gelfand-Naimark-Segal

*C** algebras as noncommutative topological spaces; von Neumann algebras as noncommutative measure theory; Type I, II, and III von Neumann algebras and its geometric meaning
(pseudo)differential operators on compact manifolds and the Atiyah-Singer index theorem
topological K-theory for Banach and C*-algebras, K-homology and KK-theory.

4 Geometry and Topology

What has come to be known as noncommutative geometry is on closer inspection really a mixture of rather independent areas which share the notion of a noncommutative space at their core. Thus within noncommutative geometry we can talk about noncommutative algebraic and differential topology, noncommutative differential geometry, etc. With this in mind it goes without saying that a practitioner of NCG should gain reasonable familiarity with the classical counterparts of these subjects.

- *Algebraic Topology*

A good introductory book on algebraic topology is Hatcher's [21]. Bott and Tu [5] offer an approach via differential forms which is specially welcome for noncommutative geometry. A history of algebraic and differential topology by Dieudonné [15] is also a good source of information.

homotopy equivalence, fundamental group and ways to compute it: covering spaces

higher homotopy groups and the fibration homotopy exact sequence

singular homology, homotopy invariance and excision; Eilenberg-Steenrod axioms

Poincaré duality and the Thom isomorphism

characteristic classes of vector and principal bundles (Chern, Pontryagin and Euler classes; Chern character)

classifying spaces of topological groups,

topological K-theory, Bott periodicity,

- *Differential Geometry*

Here is a simple example that clearly indicates why a good grasp of differen-

tial geometry is so vital for NCG: the concept of metric in NCG is defined using a Dirac operator (see Alain Connes' book [9] for explanation). In fact a key idea in NCG is to import *spectral methods* from differential geometry and analysis to the noncumulative realm. I give another example: there are many ways to define topological invariants and characteristic classes in the classical case, but only two of them, de Rham and Chern-Weil theory, extend to noncommutative geometry. For a modern survey of many aspects of differential geometry, and specially spectral geometry, check out M. Berger's *A Panoramic View of Riemannian Geometry* [3]. Modern classics of the subject include Kobayashi and Nomizu [23] and Spivak's five volume set [35]. For Dirac operators and index theory, start with Lawson and Michelson's spin geometry [24] and Roe [34]. Milnor and Stasheff [30] (specially its appendix 3) is a good place to start learning about characteristic classes and Chern-Weil theory. You will also learn a lot by consulting Atiyah's and Bott's collected works as well as Connes papers and his expository articles. ¹

smooth manifolds, differential and integral calculus on manifolds (tensor analysis), Riemannian metrics, connection and curvature
spectral geometry: Weyl's law
Chern-Weil approach to characteristic classes
index theory of elliptic operators

5 Physics

The relation between noncommutative geometry and the “mathematics of spaces” is in many ways similar to the relation between quantum physics and classical physics. One moves from the commutative algebra of functions on a space (or a commutative algebra of classical observable in classical physics) to a noncommutative algebra representing a noncommutative space (or a noncommutative algebra of quantum observable in quantum physics). Beyond this general remark, one should also bear in mind quite serious interactions between NCG and physics which include: Connes' geometrization of the standard model via noncommutative geometry², applications of NCG

¹Many of Connes' writings on the subject are now available at his website www.alainconnes.org.

²see “Noncommutative Geometry and the standard model with neutrino mixing”, hep-th/0608226, by Alain Connes, for a very recent exciting development on this

to renormalization schemes in quantum field theory (the work of Connes-Kreimer and Connes-Marcoli reported in detail in [12]), the work of Belisard in solid state physics and the quantum Hall effect, and the impact of noncommutative gauge theory on string theory. The holy grail of modern particle physics is *quantum gravity* and the ultimate unification of forces of nature at high energies. It is widely believed that NCG and a radical rethinking of the structure of spacetime will be quite relevant in this quest. It is with these connections in mind that a practitioner of NCG should pursue an understanding of modern physics.

For a general historical account of modern physics, start with the two books by Abraham Pais: *Subtle is the Lord* and *Inward Bound: Of Matter and Forces in the Physical World*. The first is a detailed account of Einstein's achievements in physics and the second is a history of elementary particle physics. I should also recommend Roger Penrose's book, *The Road to Reality: A Complete Guide to the Laws of the Universe*, which is a unique account with more details than we are used to see in such books of the current state of our understanding of high energy physics and the quest for quantum gravity. A somewhat more serious text, but with a more modest scope, is Malcolm Longair's book *Theoretical Concepts in Physics: An Alternative View of Theoretical Reasoning in Physics* [26].

- *Classical field theory*

This is essentially the theory underlying the classical physics and includes: classical mechanics, classical electrodynamics, special and general relativity, classical gauge theory, thermodynamics and classical statistical physics. A *classical field* is, at least locally, nothing but a map from a manifold M (the background space time) to a target manifold N which describes internal degrees of freedom of the field. Dynamics of the system is governed by certain partial differential equations derived from variational principles (e.g. from Lagrange's principle of least action). Among all field theories, *classical gauge theory* is the most favored tool used to probe elementary particles at a classical level. Its mathematical structure is amazingly similar to differential geometry of a principal G -bundle. One starts with a principal G -bundle over a background spacetime M , where G is the group of *internal symmetries* of the theory, describing internal degrees of freedom of particles and forces. One also fixes a linear representation of G , describing *matter fields* (G is $U(1)$, $SU(2)$, or $SU(3)$, for electrodynamics, weak, and strong force, respectively). Matter fields are sections of the associated bundle and *gauge fields* or *vector*

potentials, intermediating between matter field as forces, are connections on this bundle. The field strength, or force, is measured by the curvature of this connection. This is a the paradigm one should keep in mind when dealing with the standard model of elementary particles. Remarkably, more or less all aspects of this formalism extend to noncommutative geometry and play an essential role in Connes' geometrization of the standard model.

The books by Arnold [2] and, to some extent, Agricola-Friedrich [1] give a modern treatment of classical mechanics based on symplectic geometry of the phase space. Guillemin-Sternberg [20] is an excellent account of the origins of Hamilton's work in geometric optics and how it evolved (primarily in the hands of Jacobi) into the Hamiltonian approach to classical mechanics.

Gallileo's principle of relativity

Newtonian mechanics

Lagrangian mechanics: mechanics on the tangent bundle of the configuration space, the principle of least action, Euler-Lagrange equations

Hamiltonian mechanics: symplectic manifolds, phase space and its symplectic structure, Poisson manifolds, Hamilton's equation of motion, Liouville's theorem

completely integrable systems and Arnold-Liouville theorem on invariant tori symmetry of mechanical systems, Noether's theorem

Maxwell's equations and its modern geometric formulation

Special and general theory of relativity

Einstein's principle of relativity, Special theory of relativity

The equivalence principle, general theory of relativity

I can't resist quoting Arnold: "*Every mathematician knows that it is impossible to understand any elementary course in thermodynamics*"³. This is in part due to the fact that the geometric structures behind classical thermodynamics and statistical mechanics are more complicated than classical mechanics. The books by Agricola-Friedrich [1], Longair [26], and Bamberg-Sternberg [?] give a good accounts of the more elementary aspects of the subject. The goal of statistical mechanics is to derive the macroscopic laws of thermodynamics from the microscopic laws of statistical mechanics within the framework of classical mechanics (the work of Clausius, Boltzmann and

³Proceedings of the Gibbs Symposium; AMS publication

Gibbs). This is very well presented in [1]. When learning thermodynamics, make sure you understand the second law and the notion of entropy well.

- *Quantum field theory*

Quantum physics can be conveniently divided into *quantum mechanics* (QM) and *quantum field theory* (QFT). The latter is our ultimate weapon in probing nature at high energies, high velocities, and small distances. Make sure you understand QM well enough before jumping into QFT. The original Heisenberg, Dirac, Pauli, Jordan and Schrodinger papers of 1920's are reproduced in [39] and [36]. The original quantum electrodynamics (QED, the first successful QFT) papers are collected in [37].

Unlike quantum field theory, the mathematics of quantum mechanics is well understood and sound. It is essentially the theory of bounded and unbounded operators on a Hilbert space. The classics by Dirac, Landau-Lifschitz, and Feynman, are excellent introduction to the subject. For quantum statistical mechanics, from an operator algebraic perspective, see [6].

*why we need QM? black body radiation, emission and absorption spectrum
geometric optics-wave optics correspondence and classical mechanics-quantum
mechanics analogy*

states and observable in QM: Hilbert space and operators

Heisenberg and Schrodinger picture of QM

Heisenberg uncertainty principle

problem of quantization, Dirac quantization rules, no go theorems

quantum harmonic oscillator, Hydrogen atom

Poisson manifolds, deformation quantization of Poisson manifolds

symmetry in QM

spin, Bosons and Fermions, supersymmetry

quantum statistical mechanics

Quantum field theory combines the special theory of relativity with quantum mechanics. SR predicts that, as a result of mass energy relation, particles can be created and annihilated. Thus the number of particles in a quantum processes is not constant. Non-relativistic quantum mechanics has no way of dealing with this particle creation and annihilation problem. Apart from observed experimental facts, a major theoretical forerunner for QFT was Dirac's equation with its celebrated prediction of anti-matter and the process

matter + anti-matter \rightsquigarrow *photon*. Feynman's path integral approach greatly simplifies the QED and can be successfully applied to other QFT's including the standard model. For a general introduction to QFT start with A. Zee's *Quantum Field Theory in a Nutshell* [41]. For a more serious account start with Peskin and Schroeder [32].

physical basis of QFT (particle production and annihilation)
quantization of Lagrangian field theories via path integrals
QFT on a finite set of points-Gaussian integrals
evaluation of Gaussian integrals-stationary phase approximation and steepest descent methods; emergence of Feynman graphs and Feynman rules
divergences in perturbation expansion of path integrals
regularization and renormalization of QFT
an example: from the Dirac equation to QED; case study: compute the anomalous magnetic moment of the electron
the standard model: electroweak theory and quantum chromodynamics
difficulty with gravity and how to quantize it: noncommutative geometry, string theory, and other proposals

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The roots of noncommutative geometry lie in the theory of commutative Banach algebras and its connections with topology established by I.M. Gel'fand, M.A. Naimark, G.E. Shilov, B. Mazur and other mathematicians in the middle of XXth century. Their main idea was that basic notions of the topology of compact topological spaces may be translated into the language of commutative Banach algebras of continuous functions on these compacta. One of the goals of noncommutative geometry is to establish such correspondence between topology, analysis and differential geometry, on one hand and Banach algebras, o